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Equivariant embeddings of Hermitian symmetric spaces

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Abstract. We prove that equivariant, holomorphic embeddings of Hermitian symmetric spaces are totally geodesic (when the image is not of exceptional type).

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1. Introduction

Let H, G be connected semi-simple Lie groups and X_H, X_G the associated symmetric spaces. We assume that they are Hermitian. An equivariant embedding is a pair (F, f) where $F\colon H \to G$ is a homomorphism, $f\colon X_H \to X_G$ is a holomorphic map and

$$f(h \cdot x) = F(h)f(x), \quad x \in X_H, h \in H.$$

We assume that H, G have no compact factors and that f is injective. Then, as is easily checked, the kernel of F is finite. Replacing H by its image, we will also assume F injective and therefore identify H with its image in G.

Such maps have been classified by Satake [8] and Ihara [3] when X_H is *totally geodesic* in X_G. The purpose of this note is to show the following theorem.

Theorem. *Assume G has no factors of exceptional type. Then any equivariant embedding $X_H \to X_G$ is totally geodesic.*

We should emphasize the rather surprising content of this result when compared with the case of compact Hermitian symmetric spaces. If G is compact, the symmetric space X_G (assumed Hermitian) is a generalized Grassmanian. The natural maps of algebraic geometry between Grassmanians – in particular the Veronese and Segre embeddings – are holomorphic and equivariant with respect to natural maps of the associated groups. Very few are totally geodesic: in fact by duality between compact and non-compact symmetric spaces, the *totally geodesic* equivariant maps between compact spaces correspond to those between non-compact spaces, which are quite rare (see [2]). However, this result becomes more natural from the 'global' point of view, i.e., if one considers arithmetic quotients of the symmetric spaces.

Assume H, G are semi-simple groups defined over $\mathbb{Q}$, $F\colon H \to G$ is defined over $\mathbb{Q}$ and $f\colon X_H \to X_G$ is an equivariant embedding. For suitable arithmetic subgroups $\Delta \subset H(\mathbb{Q})$ and $\Gamma \subset G(\mathbb{Q})$, f defines a holomorphic map

$$g\colon S_H \to S_G,$$

where $S_H = \Delta\backslash X_H$ and $S_G = \Gamma\backslash X_G$.

In this situation, recall that S_H and S_G have a remarkable family of distinguished points, the CM-points or special points [5]. Also note that S_H, S_G are in fact algebraic varieties over $\mathbb{C}$, and that $g(S_H)$ is an algebraic subvariety of S_G by a theorem of Borel. Assume $g(S_H)$ has *one* CM-point. By using the action of $H(\mathbb{Q})$ on X_H one easily sees that it has a *dense* subset of CM-points for the complex topology. A conjecture of André [1] and Oort [7] then implies that $g(S_H)$ is a totally geodesic submanifold of S_G (and X_H is a totally geodesic submanifold of X_G).

It is not obvious that $g(S_H)$ should have one CM-point; note, however, the following. The Hermitian symmetric spaces are open subspaces of their compact duals – generalized Grassmanians. An equivariant holomorphic embedding will generally be given by a natural holomorphic map between the compact duals. Given the $\mathbb{Q}$-structure, CM-points correspond to subspaces (in the Grassmanians) verifying some rationality conditions. It is natural to expect these to be preserved. The embedding of the symmetric space for $SU(p,1)$, $X_{p,1}$, into $X_{P,Q}$ where $P = \binom{p}{k}$, $Q = \binom{p}{k-1}$ [8] gives a very graphic example.

Another strong motivation for the theorem is given by Mok's rigidity results. Assume for simplicity that H is irreducible over $\mathbb{Q}$ and $\mathrm{rk}(H) > 1$ (this is the *real* rank). Then Mok (Ch. 6, Thm 4.1 of [4]) – see also the discussion at the beginning of ch. 9 – has shown that *any* holomorphic map $S_H \to S_G$ is totally geodesic. If $F\colon H(\mathbb{R}) \to G(\mathbb{R})$ (we now denote the Lie groups by $G(\mathbb{R})$, $H(\mathbb{R})$ as we will be using rationality arguments) is given and if F is $G(\mathbb{R})$-conjugate to a map defined over $\mathbb{Q}$, Mok's theorem implies our local assertion. More generally, assume F given, and assume that there exists a totally real number field L and a map $F_L\colon H \to G$ defined over L such that, for each real prime v of L (thus $L_v \cong \mathbb{R}$),

$$F_{L,v}\colon H(\mathbb{R}) \to G(\mathbb{R})$$

is conjugate to F. Then, again using Mok's results, we deduce that F is totally geodesic.

The set of homomorphisms $F\colon H \to G$, over an algebraically closed field, and modulo G-conjugation, is *discrete* (homomorphism of semi-simple groups up to conjugacy are rigid). Thus $F\colon H(\mathbb{R}) \to G(\mathbb{R})$ is $G(\mathbb{R})$-conjugate to a map F_1 defined over $\bar{\mathbb{Q}}$; the G-conjugacy class of F_1 is an irreducible variety. If it is defined over $\mathbb{Q}$, a theorem of Moret–Bailly [6] implies that there is a totally real number field L, and a map $F_L\colon H \otimes_{\mathbb{Q}} L \to G \otimes_{\mathbb{Q}} L$ verifying our condition.

It is of course, difficult to compute the field of rationality of the class associated to F. One may, however, pose the following:

Problem. If H, G be semisimple groups over $\mathbb{Q}$ and $F\colon H \to G$ a homomorphism defined over $\mathbb{R}$, does there exist a totally real field L and $F_L\colon H \to G/L$ such that F_v is $G(\mathbb{R})$-conjugate to F at each real prime of L?

Finally, Mok has informed us that he could prove the theorem even for exceptional G. His proof, however, is more difficult and necessitates global geometric computations.

2. Reductions

Let G be a connected semi-simple Lie group, with finite center and no compact factor, associated to a Hermitian symmetric space X. Fix a point $x \in X$. Then x defines a maximal compact subgroup $K \subset G$ and a Cartan involution θ on $\mathfrak{g} = \mathrm{Lie}(G)$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition. There exists an element $\zeta \in Z(K)$ such that $\mathrm{Ad}(\zeta)$ induces on $\mathfrak{p}$ the multiplication by $i = \sqrt{-1}$ defining the complex structure. Then $\zeta^2 \in Z(K)$ induces, by the adjoint action, the Cartan involution. By construction this holomorphic structure is G-equivariant: if $x' = g \cdot x$ the associated data are obtained by conjugation by g. In particular, $\zeta' = \mathrm{Ad}(g)\zeta \in K'$ is well-defined by x' since ζ is K-invariant, and this family of quasi-complex structures defines the holomorphic structure on X.

Now assume $H \subset G$, $f\colon X_H \to X_G$ verify our conditions. Fix a base point $x \in X_H$. This defines maximal compact subgroups $K_H \subset K_G$. (We will drop indexes for the group G). Thus

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,,$$
$$\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$$

and the (injective) map $F\colon \mathfrak{h} \to \mathfrak{g}$ has the following properties:

$$F(\mathfrak{k}_H) \subset \mathfrak{k}, \tag{1}$$
$$F(X) = F_c(X) + F_p(X), \tag{2}$$
$$(X \in \mathfrak{p}_H, F_c(X) \in \mathfrak{k}\,,\ F_p(X) \in \mathfrak{p})$$
$$F_p(\iota_H X) = \iota_G\, F_p(X), \tag{3}$$

where ι_H, ι_G are 'multiplication by $\sqrt{-1}$' on $\mathfrak{p}_H$, $\mathfrak{p}$, given by ζ_H, ζ_G. Conversely, if a morphism $F\colon \mathfrak{h} \to \mathfrak{g}$ verifies (1)–(3), F defines a map $H/K_H \xrightarrow{f} G/K_G$, holomorphic at $x = eK_H$ and in fact at every point by a computation similar to that as above. Note that f is a totally geodesic immersion if and only if,

$$F(\mathfrak{p}_H) \subset \mathfrak{p}\ \text{, i.e., if } F_c \equiv 0.$$

(see p. 47 ff. of [8])

Identifying $\mathfrak{k}_H$ with a subalgebra of $\mathfrak{k}$ by (1), we note that the two components F_c and F_p are $\mathfrak{k}_H$-equivariant. Moreover, let $\mathfrak{h} = \oplus \mathfrak{h}_i$ be a decomposition of $\mathfrak{h}$ in simple factors. Then ζ_H or ι_H decomposes accordingly, so the restriction F_i to $\mathfrak{h}_i$ again verifies the conditions. Thus we may assume that $\mathfrak{h}$ is simple.

In this case it is known (see S Helgason, Differential Geometry and Symmetric Spaces, ch. VIII, §5) that the (real) representation of $\mathfrak{k}_H$ on $\mathfrak{p}_H$ is irreducible. The $\mathfrak{k}_H$-map $F_c\colon \mathfrak{p}_H \to \mathfrak{k}$ is therefore injective or zero. Assume (changing notation) that $\mathfrak{h}_1 \subset \mathfrak{h}$ is a θ-stable semi-simple subalgebra such that the injection $\mathfrak{p}_1 \subset \mathfrak{p}_H$ is holomorphic (for the choice of $\zeta_1 \in Z(K_1)$ where K_1 is the obvious maximal compact subgroup of $H_1 = \exp(\mathfrak{h}_1) \subset H$).

It suffices then to check that $F_c = 0$ on $\mathfrak{p}_1$. But any Hermitian Lie algebra $\mathfrak{h}$ contains a subalgebra $\mathfrak{h}_1$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, the injection being holomorphic in the obvious sense (in fact it contains $\mathfrak{sl}(2,\mathbb{R})^r$ where r is the real rank (see e.g. Ch. 5 of [4]). Thus we are reduced to the case when $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{R})$.

We can also replace G by a larger group. By the results of Satake, X_G embeds into X_{G_1} where $G_1 = SU(p,p)$, as a totally geodesic subvariety, via an equivariant embedding. Finally we are reduced to the case when H is locally isomorphic to $SL(2,\mathbb{R})$ or $SU(1,1)$ and G to $SU(p,p)$. (Note that this does not apply when G has exceptional factors).

3. Computations

In this paragraph we consider the case, to which we are reduced, when $H = SU(1,1)$ and $G = SU(p,p)$. We try to solve the linear algebra problem of § 2 – find F verifying (1)–(3). We have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & z \\ \overline{z} & -a \end{pmatrix} : z \in \mathbb{C}\ ,\ a \subset i\mathbb{R} \right\}, \tag{4}$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & Z \\ {}^t\overline{Z} & B \end{pmatrix} : \mathrm{Tr}(A) + \mathrm{Tr}(B) = 0 \right\}, \tag{5}$$

where the block matrices are of size $p \times p$, Z is (complex) arbitrary and A, B are skew-hermitian. Let $u = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} & i \\ -i & \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)$, a basis of $\mathfrak{h}$ (the empty entries are zero). Let

$$x = \begin{pmatrix} & 1 \\ & \end{pmatrix} \quad y = \begin{pmatrix} & \\ 1 & \end{pmatrix}\ ,\ h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\ ,$$

a basis of $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$. We take $\mathfrak{k} \subset \mathfrak{g}$ given by block-diagonal matrices, so

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\ ,\ \mathfrak{p} = \left\{ \begin{pmatrix} & Z \\ Z^* & \end{pmatrix}, Z \in M_p(\mathbb{C}) \right\},$$

where $Z^* = {}^t\overline{Z}$. Similarly,

$$\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H\ ,\ \mathfrak{k}_H = \mathbb{R}w\ ,\ \mathfrak{p}_H = \mathbb{R}v + \mathbb{R}w.$$

If $F\colon \mathfrak{h} \to \mathfrak{g}$ verifies (3) we have

$$F(u) = \begin{pmatrix} A & Z \\ Z^* & B \end{pmatrix}, \tag{6}$$

$$F(v) = \begin{pmatrix} C & iZ \\ -iZ^* & D \end{pmatrix}, \tag{7}$$

$A, \ldots, D$ verifying of course (5). Let X, Y, H be the images of x, y, h. Using (5), (6) and (7) we have

$$X = \begin{pmatrix} E & Z \\ & F \end{pmatrix}, \tag{8}$$

$$Y = \begin{pmatrix} -E^* & \\ Z^* & -F^* \end{pmatrix}, \tag{9}$$

$$H = [X,Y] = \begin{pmatrix} -[E,E^*] + ZZ^* & -ZF^* + E^*Z \\ FZ^* - Z^*E & -[F,F^*] - Z^*Z \end{pmatrix}, \tag{10}$$

where E, F and Z are arbitrary $p \times p$-matrices (with $\mathrm{Tr}(E) + \mathrm{Tr}(F) = 0$). Since $h = i^{-1}w$, H is block-diagonal by (1); conjugating w under $K = S(U(p) \times U(p))$ we can assume that the block-diagonal entries of H are diagonal matrices H_1, H_2. The eigenvalues of H are integral, and constitute the eigenvalues of a representation of $\mathfrak{sl}(2,\mathbb{C})$.

Let $V \cong \mathbb{C}^{2p}$ be the space of the natural representation of G, and $V = V_+ \oplus V_-$ its decomposition into a positive and a negative subspace. Then

$$E : V_+ \to V_+,$$
$$F : V_- \to V_-,$$
$$Z : V_- \to V_+.$$

Let $\lambda_1 > \cdots > \lambda_{t+1}$ be the distinct eigenvalues of H in V_+ and $\mu_1 > \cdots > \mu_{s+1}$ the eigenvalues in $V_- : s,t \geq 0$. We can write $V = V^{\text{even}} \oplus V^{\text{odd}}$, the eigenvalues being even or odd in each summand; this decomposition is preserved by X and Y. The decomposition is orthogonal and compatible with $V = V_+ \oplus V_-$. If v belongs to the λ-eigenspace of V_+ (resp. V_-), Ev (resp. Zv, Fv) belongs to the $(\lambda+2)$-eigenspace of V_+ (resp. V_+, V_-)

Consider first the odd part of V. We can write in V_+^{odd}:

$$E = \begin{pmatrix} 0 & E_1 & & & \\ & 0 & E_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & E_t \\ & & & & 0 \end{pmatrix}, \quad E^* = \begin{pmatrix} 0 & & & \\ E_1^* & \ddots & & \\ & \ddots & \ddots & \\ & & E_t^* & 0 \end{pmatrix}.$$

Writing $\operatorname{diag}(A_1, \ldots, A_{t+1})$ for a *block*-diagonal matrix we have

$$EE^* = \operatorname{diag}(E_1E_1^*, \ldots, E_tE_t^*, 0)$$
$$E^*E = \operatorname{diag}(0, E_1^*E_1, \ldots, E_t^*E_t).$$

According to (10),

$$\begin{aligned} -[E,E^*] + ZZ^* &= \operatorname{diag}(-E_1E_1^*, E_1^*E_1 - E_2E_2^*, \ldots, E_t^*E_t) + ZZ^* \\ &= \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{t+1}), \end{aligned} \tag{11}$$

where the eigenvalues are now those in V_+^{odd}, the last 'diagonal' matrix including of course the multiplicities. Considering the restriction of the corresponding Hermitian forms to the last summand we see that

$$E_tE_t^* + ZZ^* = \lambda_{t+1} \geqslant 0;$$

since the representation is odd, $\lambda_1 > \cdots > \lambda_{t+1} > 0$.

Similarly in V_-^{odd}:

$$F = \begin{pmatrix} 0 & F_1 & & & \\ & 0 & F_2 & & \\ & & 0 & & \\ & & & \ddots & F_s \\ & & & & 0 \end{pmatrix}, \quad F^* = \begin{pmatrix} 0 & & & \\ F_1^* & 0 & & \\ & & \ddots & \\ & & F_s^* & 0 \end{pmatrix},$$

$$\begin{aligned} -[F,F^*] - Z^*Z &= \operatorname{diag}(-F_1F_1^*, \ldots, F_s^*F_s - Z^*Z \\ &= \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{s+1}) \end{aligned} \tag{12}$$

whence $0 > \mu_1 > \cdots > \mu_{s+1}$.

Finally the only non-vanishing part of Z is a map $Z_1 \colon V_-(-1) \to V_+(1)$ (where $V(\lambda)$, $V_\pm(\lambda)$ denote the eigenspaces of H). Thus

$$ZZ^* = \operatorname{diag}(0, 0, \ldots, Z_1 Z_1^*), \tag{13}$$

$$Z^* Z = \operatorname{diag}(Z_1^* Z_1, 0, \ldots, 0). \tag{14}$$

By (11) and (13),

$$\operatorname{diag}(-E_1 E_1^*, E_1^* E_1 - E_2 E_2^*, \ldots, E_t^* E_t + Z_1 Z_1^*) = (\lambda_1, \ldots, \lambda_{t+1}) \tag{15}$$

with positive eigenvalues. This is impossible unless

$$\begin{cases} t = 0,\ \lambda_1 = 1,\ E = 0\ , \\ Z = Z_1,\ ZZ^* = 1\ . \end{cases} \tag{16}$$

(The identity (15) implies that $t = 0$; since there is only one eigenvalue, the representation theory of $SL(2)$ forces it to be 1.)

This implies of course that the only eigenvalue μ is -1, so $s = 0$ and $F = 0$. Since E, F vanish the embedding is totally geodesic; the representation of $SL(2)$ or $SU(1,1)$ is a multiple of the standard representation, in conformity with Satake's results.

Consider now the even part of V. The first part of the argument still applies, yielding now

$$\lambda_1 > \cdots > \lambda_{t+1} \geqslant 0, \tag{17}$$

$$0 \geqslant \mu_1 > \cdots > \mu_{s+1}. \tag{18}$$

Now Z is the sum of

$$Z_1 : V_-(0) \to V_+(2),$$

$$Z_2 : V_-(-2) \to V_+(0).$$

Thus

$$ZZ^* = \operatorname{diag}(0, 0, \ldots, 0, Z_1 Z_1^*, Z_2 Z_2^*), \tag{19}$$

$$Z^* Z = \operatorname{diag}(Z_1^*, Z_1, Z_2^* Z_2, 0, \ldots, 0). \tag{20}$$

By (11) and (19),

$$\begin{aligned} &-[E_1 E^*] + ZZ^* \\ &\quad = (-E_1 E_1^*, E_1^* E_1 - E_2 E_2^*, \ldots, E_{t-1}^* E_{t-1} - E_t E_t^* + Z_1 Z_1^*, E_t^* E_t + Z_2 Z_2^*) \\ &\quad = (\lambda_1, \ldots, \lambda_t, \lambda_{t+1}), \end{aligned} \tag{21}$$

where we assume so far that both 2 and 0 are eigenvalues in V_+^{even}. This implies first that there are only two eigenvalues since $-E_1 E_1^* = \lambda_1 > 0$ for $t > 1$. Furthermore, the last entry in (21) yields $E_1 E_1^* + Z_2 Z_2^* = 0$, whence $E = E_1 = 0$ and $Z_2 = 0$.

If 2 does not occur in V_+^{even}, the representation on V^{even} is trivial; if 0 does not occur Z_2 is absent. In this case,

$$ZZ^* = \operatorname{diag}(0, \ldots 0, Z_1 Z_1^*),$$

$$Z^* Z = \operatorname{diag}(Z_1^* Z_1, 0, \ldots 0) \tag{22}$$

and

$$-[E, E^*] + ZZ^* = (-E_1E_1^*, E_1^*E_1 - E_2E_2^*, \ldots, E_t^*E_t + Z_1Z_1^*)$$
$$= (\lambda_1, \ldots, \lambda_{t+1})$$

with $\lambda_{t+1} = 2$. This equality $(-E_1E_1^* = \lambda_1 > 0)$ implies that there is only one eigenvalue $(t = 0)$ and therefore $E = 0$.

Of course, a similar computation, as in the odd case, applies to the negative part, using now (20); if there are two eigenvalues $(0, -2)$ we deduce that

$$F = F_1 = 0\ ,\ Z_1 = 0.$$

Thus $Z = 0$, contrary to the assumption that it represented the tangent map to an equivariant embedding.

Finally, consider the case where 0 does not occur in V_+^{even} or V_-^{even}.

The computations being symmetric we can assume for instance that it is missing in V_+^{even}; we already know that the eigenvalue 2 only occurs, so the eigenvalues in V^{even} are $(2, 0, -2)$; moreover $E = F = 0$ by the arguments given already, so the embedding should be totally geodesic. We know that this is impossible, by Satake's results. In fact, using (22) and (12) we see that

$$\mathrm{diag}(-F_1F_1^* - Z_1Z_1^*, F_1^*F_1) = (\mu_1, \mu_2) = (0, -2)$$

which is impossible.

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